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Composite Variational Principles, Added Variables, and Constants of Motion

Giacomo Caviglia[1]

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It is shown that any second-order differential system admits a variational formulation via the introduction of suitable additional variables. The new variables are related to the existence of invariant 1-forms and to solutions for the adjoint of the equations of variation of the given system. The connections among invariant forms, constants of motion, and infinitesimal invariance transformations are then discussed in some detail.

1. INTRODUCTION

Consider a system of n second-order ordinary differential equations of the form

$$\ddot{q}^i - F^i(t, q, \dot{q}) = 0, \qquad i = 1, \ldots, n \tag{1}$$

where $\dot{}$ denotes as usual the total d/dt derivative and each F^i is supposed to be a regular function of its arguments. As is well known, the system (1) admits a variational formulation only if it is self-adjoint (Santilli, 1978; Crampin, 1981). This means that the functions F^i satisfy a set of quite stringent conditions, which have been extensively analyzed in a number of papers (see, e.g., Santilli, 1978; Caviglia, 1985b and the cited references).

Nevertheless, Lagrangian representations of some non-self-adjoint systems have already been developed (Morse et al., 1953; Tikochinsky, 1978; Greenberger, 1979); they are based on a trick consisting essentially in the introduction of additional variables that are related to the q's by a suitable set of differential equations. In this way a Lagrangian function yielding both the original system and the added equations is found, at the sacrifice of a certain amount of "reality" in some of the incident results, such as, e.g., the introduction of an oscillator with negative friction in a specific case (Morse et al., 1953).

[1]Istituto Matematico dell'Università, Via L. B. Alberti 4, 16132 Genova, Italy.

It is shown in this work that any system (1) may be derived from a variational principle of the above type (Section 2). The possible meaning of the added variables is then discussed in some detail. In particular, the geometric origin of these variables as generators of invariant forms is brought into evidence, and it is also shown that they are related to the symmetries of the given dynamical system in such a way that allows for extension to infinite dimensions (Section 3). The resulting role of the new variables as generators of conserved quantities is also examined (Section 4).

2. VARIATIONAL PRINCIPLES

Consider the following action functional:

$$J(q, \xi) = \int_{t_0}^{t_1} (\dot{q}^i \dot{\xi}_i + F^i \xi_i)\, dt \tag{2}$$

The requirement of stationarity for J under variations of the q's and of the ξ's leads to equations

$$\ddot{\xi}_h + \left(\xi_i \frac{\partial F^i}{\partial \dot{q}^h} \right)^{\cdot} - \xi_i \frac{\partial F^i}{\partial q^h} = 0 \tag{3}$$

and (1), respectively. In particular, the system (1), (3) is self-adjoint, because it has been derived from a variational principle (Santilli, 1978). Thus we may also assert that when equations (3) are joined to the original set (1) we obtain a second-order system satisfying the conditions for the existence of a formulation in terms of Hamilton's principle.

Actually, the functional J cannot be regarded as providing a variational formulation for (1) in a strict sense, because it requires the introduction of the auxiliary variables ξ. Yet it is to be remarked that the possibility of finding variational principles by addition of new variables has been rather explored in the literature (Hojman et al., 1981; Leipholz, 1980; Morse et al., 1953; Tikochinsky, 1978; Greenberger, 1979; Thangaray et al., 1983); in particular, linear systems have been extensively analyzed by Bahar and Kwatny (1984), whereas it turns out that the functional J may be reduced to the form of a so-called composite variational principle (Atherton et al., 1975) by means of an integration by parts.

The present investigation is mainly devoted to the discussion of another major problem which arises within the present framework: specifically, it is concerned with the meaning to be attributed to the auxiliary variables ξ, which are usually introduced on rather unnatural and purely formal grounds. The following examples will clarify this point.

Consider the one-dimensional oscillator with friction, having the equation of motion

$$\ddot{q} + \varepsilon\dot{q} + \nu q = 0 \tag{4}$$

Then the action functional J reads

$$J(q, \xi) = \int_{t_0}^{t_1} [\dot{q}\dot{\xi} - (\varepsilon\dot{q} + \nu q)\xi]\, dt \tag{5}$$

whereas (3) reduces to

$$\ddot{\xi} - \varepsilon\dot{\xi} + \nu\xi = 0 \tag{6}$$

so that the added variable ξ has been interpreted as representing the evolution of a "mirror image" oscillator with "negative" friction (Morse et al., 1953).

As a second illustrative example, consider the equation for the geodesics of a linear symmetric connection, which is known to model the evolution of holonomic rheonomic systems in n degrees of freedom (Trumper, 1983). This equation reads

$$\ddot{q}^i + \Gamma^i_{jk}\dot{q}^j\dot{q}^k = 0 \tag{7}$$

where the Γ^i_{jk}'s are the connection coefficients and t is any affine parameter; the corresponding action functional can be cast into the form

$$J(q, \xi) = \int_{t_0}^{t_1} (\dot{q}^i\dot{\xi}_i - \Gamma^i_{jk}\dot{q}^j\dot{q}^k\xi_i)\, dt = \int_{t_0}^{t_1} \dot{q}^i D\xi_i/Dt\, dt \tag{8}$$

yielding the variational principle for the geodesics of the given connection already introduced by Trumper (1980). In this case the interpretation of the added variables is more natural, in the sense that they may be reduced to already known geometric objects. In fact (3) may be written as an equation of geodesic deviation, namely,

$$D^2\xi_h/Dt^2 + R^p_{ihj}\xi_p\dot{q}^i\dot{q}^j = 0 \tag{9}$$

where R is the curvature tensor of the given connection. Thus the auxiliary variables identify a Jacobi covector (Schattner et al., 1981) as well as the related first integral (Trumper, 1983)

$$\dot{q}^k D\xi_k/Dt = \dot{q}^k(\dot{\xi}_k - \Gamma^p_{kh}\xi_p\dot{q}^h) \tag{10}$$

which is essentially the Jacobi integral for the dynamical system modeled by the action functional (8).

Now it is to be recalled that Jacobi covectors are strictly connected to the description of invariance properties of the geodesic equation (Caviglia, 1983a,b); this observation suggests the most appropriate setting for the study of added variables, and motivates the analysis which is given in the following section.

3. ADJOINT VARIABLES

The system (1) may be associated with the vector field

$$\Gamma = \frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial q^i} + F^i \frac{\partial}{\partial \dot{q}^i} \tag{11}$$

of the $(2n+1)$-dimensional extended tangent space E, referred to local natural coordinates $(t, q, \dot{q})$: the flow of Γ yields the totality of solutions to (1) and conversely (Sarlet et al., 1981).

Consider now a 1-form α of E with local expression

$$\alpha = \lambda\, dt + \eta_i\, dq^i + \xi_i\, d\dot{q}^i \tag{12}$$

and denote by $\mathscr{L}_\Gamma$ the Lie differentiation operator along Γ. Making use of (11) and (12) it follows that the condition

$$\mathscr{L}_\Gamma \alpha = 0 \tag{13}$$

is equivalent to

$$\dot{\lambda} + \xi_i \frac{\partial F^i}{\partial t} = 0 \tag{14a}$$

$$\dot{\eta}_j + \xi_i \frac{\partial F^i}{\partial q^j} = 0 \tag{14b}$$

$$\dot{\xi}_j + \eta_j + \xi_i \frac{\partial F^i}{\partial \dot{q}^j} = 0 \tag{14c}$$

Then it is easily seen that the restriction of the components $\xi_i(t, q, \dot{q})$ to any integral curve of Γ yields a solution to the adjoint system (3), provided (13) holds. In fact (14b) reduces to (3)—up to a change in sign—after substitution of the expression for η_j obtained from (14c). Conversely, each set of functions $\xi_i(t, q, \dot{q})$ fulfilling (3) on every solution to (1) identifies an invariant 1-form α. Specifically, (14b) is automatically satisfied, provided η_j is defined in terms of ξ_j through (14c), and λ is determined as a solution to (14a). Notice that it is not restrictive to set $\lambda = 0$ whenever the F^i's are independent of t.

In summary, every 1-form which is invariant along the flow of Γ can be made to correspond to a set of added variables, and conversely. When the special case of the geodesic equation is considered, it follows in particular that Jacobi covectors can be looked at as invariant 1-forms of E, and conversely. Consequently, it could be shown that invariant 1-forms depending polynomially on $\dot{q}^h$ can be associated with the Killing tensors of the given linear connection, by slightly modifying the proof of similar statements holding for dynamical symmetries (Caviglia, 1983a,b; Prince et al., 1984).

In order to describe an alternative interpretation for the ξ's we need the concept of dynamical symmetry, defined as a vector field $\bar{Y}$ on E which is the infinitesimal generator of a local 1-parameter group of transformations permuting the integral curves of Γ, i.e., such that $\mathcal{L}_{\bar{Y}}\Gamma$ is proportional to Γ. Clearly, dynamical symmetries are defined up to a multiple of Γ (Sarlet et al., 1981). Thus it follows that if $\bar{Y}$ is any dynamical symmetry, then $Y=\bar{Y}-\tau\Gamma$ is an equivalent dynamical symmetry; therefore, if we choose as τ the $\partial/\partial t$ component of $\bar{Y}$, then Y may be represented as

$$Y=K^i\frac{\partial}{\partial q^i}+H^i\frac{\partial}{\partial \dot{q}^i} \tag{15}$$

and we get

$$\mathcal{L}_Y\Gamma=0 \tag{16}$$

Moreover, on writing the explicit expression for $\mathcal{L}_Y\Gamma$ it is found that (16) is equivalent to

$$H^i=\dot{K}^i \tag{17}$$

$$M^i(K)=-\frac{\partial F^i}{\partial q^h}K^h-\frac{\partial F^i}{\partial \dot{q}^h}\dot{K}^h+\ddot{K}^i=0 \tag{18}$$

In particular, (17) may be regarded as the definition of H^i in terms of K^i, so that (18)—which has been written taking into account (17)—yields the characterization of the dynamical symmetry Y.

Alternatively, it may be shown that the infinitesimal perturbation

$$\bar{q}^i=q^i+\varepsilon K^i(t,q,\dot{q}) \tag{19}$$

leaves equations (1) invariant—up to terms of order ε^2—iff the generators K^i satisfy condition (18) (Caviglia, 1985a). In this sense, the definition of dynamical symmetry coincides with the so-called equations of variation for (1) (Santilli, 1978), and the components K^i may be regarded as generators of infinitesimal symmetry transformations.

Consider now the adjoint to $M^i(K)$, say $\tilde{M}_i(\xi)$, which is defined by the condition (Santilli, 1978)

$$\xi_i M^i(K)-K^i\tilde{M}_i(\xi)=\left(-\xi_i\frac{\partial F^i}{\partial \dot{q}^h}K^h+\xi_i\dot{K}^i-\dot{\xi}_iK^i\right)^{\cdot} \tag{20}$$

On comparing (18) and (3) it is easily seen that $\tilde{M}_i(\xi)$ does coincide with the left-hand side of equation (3). Accordingly, we conclude that the system (3) is formed by the adjoint equations to the equations of variation for (1)—or to the definition of dynamical symmetry—in the technical sense of the term adjoint. Of course, equations (3) coincide with the adjoint equations of (1) under the additional assumption that the given system is linear, in which case our analysis reduces to already known results (Bahar et al., 1984).

As a comment, let us firstly point out explicitly that our approach has also shown the connections between invariant forms and solutions to the adjoint of the equations of variations. Secondly, the formulation in terms of infinitesimal perturbations can be extended straightforwardly to composite variational principles of nonlinear field theories, where, of course, equation (20) is to be replaced by the so-called Lagrange identity. This will be the subject of a forthcoming paper. Thirdly, the previous interpretation relating the adjoint variables ξ to invariant forms brings into evidence their role as generators of conservation laws (Caviglia, 1984), which is examined in the next section.

4. CONSERVATION LAWS AND FINAL COMMENTS

The previous interpretations of the meaning of the additional variables ξ give rise to alternative approaches to the problem of generating conservation laws. Perhaps, the geometric formulation is simpler, so that we will discuss it in some detail.

It is an immediate consequence of (13) that

$$I_1 = \langle \Gamma, \alpha \rangle = \lambda + \dot{q}^i \eta_i + F^i \xi_i \tag{21}$$

is conserved on each solution of (1), since $\mathscr{L}_\Gamma I_1 = 0$. To comment on the meaning of I_1 it is to be noticed that I_1 reduces to the opposite of the first integral (10) in the specific case of geodesic motion. Also important for practical purposes is the observation that I_1 is written entirely in terms of the given invariant form α. In this respect, the approach to constants of motion by means of invariant forms should be regarded as particularly worth trying. To appreciate its effectiveness, it suffices to note that commonly proposed methods for associating constants of motion with dynamical symmetries require either the solution of an additional differential equation, when they are based on Noether theorem (Sarlet et al., 1981), or the introduction of stringent assumptions such as: there exists an invariant volume form (Crampin, 1980), there exist suitable differential forms (Gonzalez-Gascon et al., 1980), equations (1) may be derived from a variational principle (Caviglia, 1984).

Further constants of motion can be associated with invariant 1-forms provided at least one symmetry transformation is given. Actually, it follows from (16) and the properties of the Lie derivative that the quantity

$$I_2 = \langle Y, \alpha \rangle = \dot{K}^i \xi_i - K^i \left(\dot{\xi}_i + \xi_h \frac{\partial F^h}{\partial \dot{q}^i} \right) \tag{22}$$

is a constant motion. If the equation of motion is the geodesic equation (7) then Y identifies a Jacobi vector on each geodesic (Caviglia, 1983a,b; Prince et al., 1984), and I_2 can be shown to coincide with the already known

(Trumper, 1983; Caviglia, 1983a) conserved quantity $\xi_i DK^i/Dt - K^i D\xi_i/Dt$.

Turning to general considerations, it is to be remarked that the previous discussion can be reinterpreted in terms of Crampin's (1983) approach to non-Noether constants of motion by noting that Y and α are nothing but the generators of a type $(1,1)$ tensor field $\Lambda = Y \otimes \alpha$ such that $\mathscr{L}_\Gamma \Lambda = 0$. Then the trace of Λ and of its powers, i.e., I_2 and its powers, are conserved along any integral curve of Γ. Moreover, we may also recall that $(1,1)$ tensors with vanishing Lie derivative along Γ give rise to a Lax pair (Carinena et al., 1985), which in turn is related to the complete integrability of the given system (De Filippo et al., 1984; Antonini et al., 1985), to infer the existence of relationships between the definition (16) of dynamical symmetry and its adjoint (3) on the one hand, and the aforementioned complete integrability on the other, which should deserve further investigation.

To conclude, we would like to emphasize the following point. If the approach to conservation laws is based on the fact that the system (3) is the adjoint system of (18), then the Lagrange identity (20) yields immediately the first integral I_2. On the contrary, the determination of I_1 is not so straightforward because I_1 depends both on the ξ's and on λ, the latter being determined as a solution to (14a). Nevertheless, the construction of I_1 may be achieved without having explicit recourse to invariant forms, by manipulations of the equation which is found by contraction of (3) with $\dot{q}^h$, provided a solution to (14a) is given. Therefore it could seem that the approach to the added variables in terms of adjoint equations is not convenient and may be rejected in favor of the equivalent geometric formulation, which could also benefit from a great number of recent developments in the field. However it is to be emphasized that just the former approach can be extended quite straightforwardly to continuum mechanics and field theories, as it will be shown in detail in a forthcoming paper. Thus an analysis of the relationships between the two formulations is required to suggest the most appropriate procedures for the construction of conservation laws within the framework of field theories.

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